2.1 Tangent and Velocity Problems

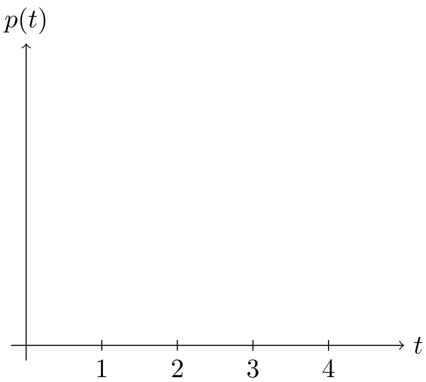
Learning Objectives: After completing this section, we should be able to

- approximate the slope of the tangent line to a curve at a point.
- approximate the instantaneous velocity of a moving object at a particular moment.

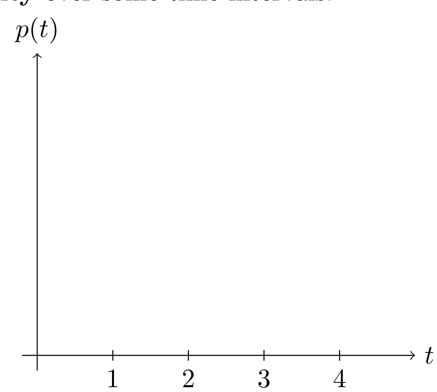
Driving question to start: If we know the exact position of on object, how can we find its velocity?

2.1.1 Limits

Example. Suppose we throw a baseball into the air. The function $p(t) = 64t - 16t^2$ gives the ball's height in feet at any time t seconds after throwing it. What is the velocity at $t = 1$ seconds?
Let's start with a graph:



Can we first approximate the velocity? Let's find the **average velocity** over some time intervals.



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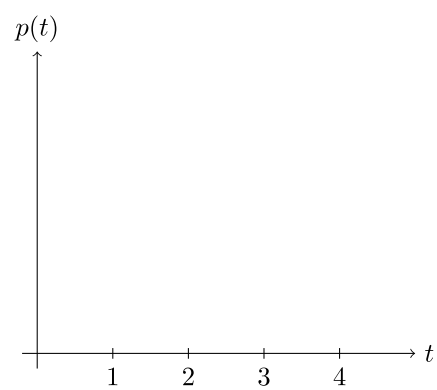
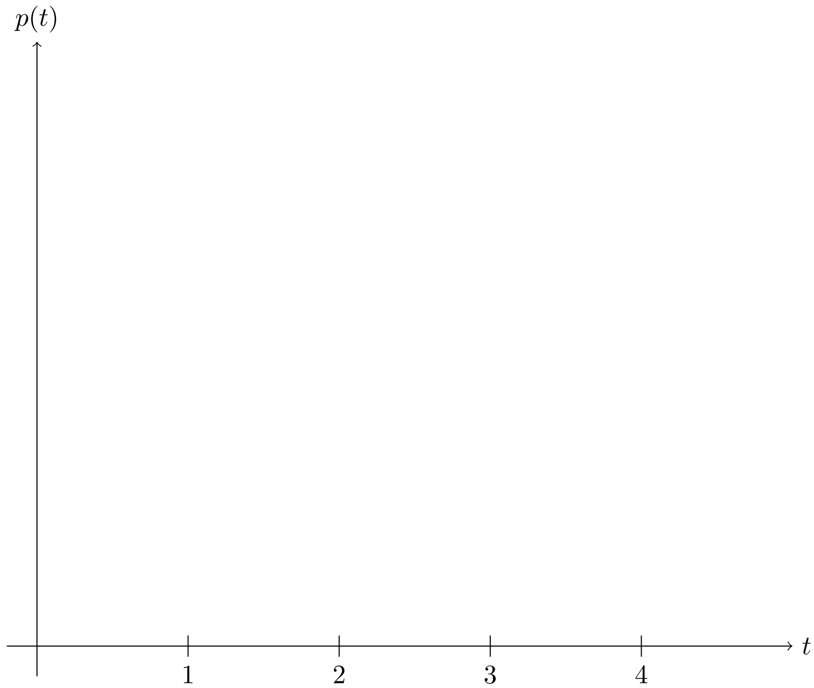


So we have done several approximations. What is the end goal?



Definition. *Instantaneous Velocity* is the slope of

As the second time t is closer to $t = 1$ in our approximations, the average velocity

This is a limit! The *limit* as t approaches 1 of the

2.2 The Limit of a Function

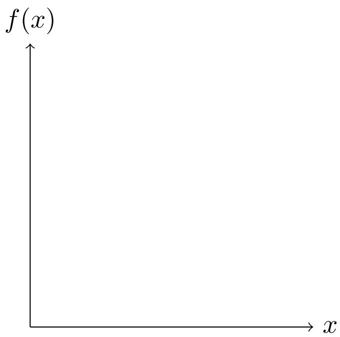
Learning Objectives: After completing this section, we should be able to

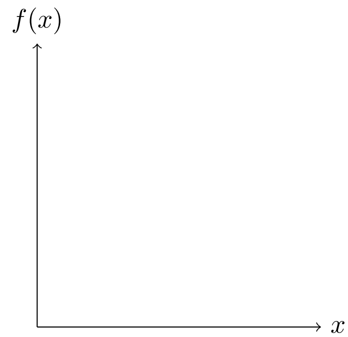
- define the limit of a function and make educated guesses at limits.
- define the one-sided limit of a function and make educated guesses at limits.

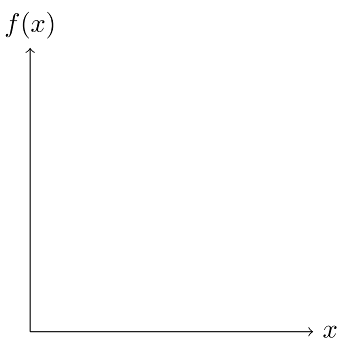
2.2.1 Limit Definition

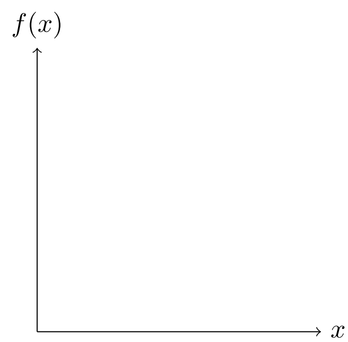
Definition. $\lim_{x \to a} f(x) = L$ means

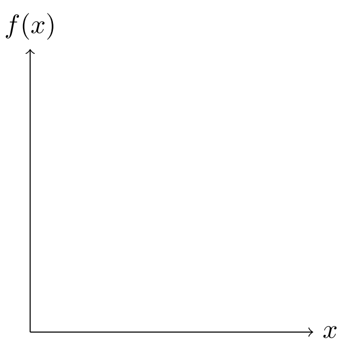
Let's look at several examples:

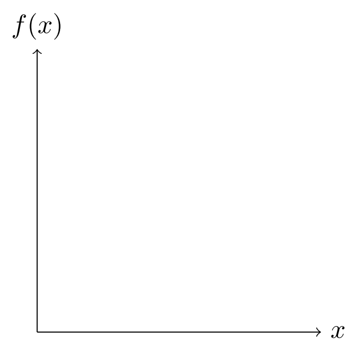












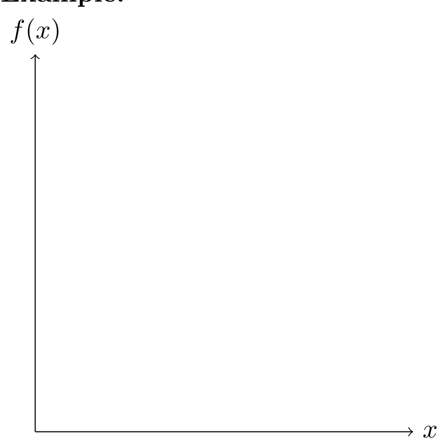
Note, for $\lim_{x \to a} f(x) = L$, $f(x)$ must be arbitrarily close to L for

Definition. *(One sided limit)* If $f(x)$ is arbitrary close to L for all

Definition. *(Right-hand limit)*

Definition. *(Left-hand limit)*

Example.



Example. Let

$$f(x) = \begin{cases} 2x+1, & x > 1, \\ 2x, & x < 1. \end{cases}$$

2.2.2 Indeterminate Forms

Question. *What could happen for a function $f(x)$ to **NOT** have a limit?*

Example.

$f(x)$	$g(x)$	$h(x)$
$\longleftrightarrow x$	$\longleftrightarrow x$	$\longleftrightarrow x$

Question. *How can we recognize these examples from the functions $f(x)$, $g(x)$, and $h(x)$?*

In general, if $f(x)$ has *bad behavior* at $x = a$, then $\lim_{x \to a} f(x)$ may not exist.

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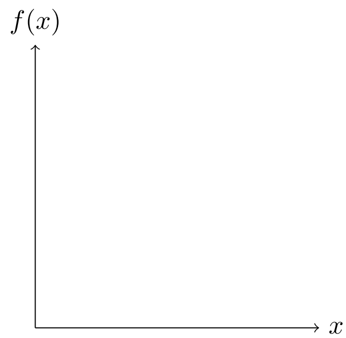
-
-
-
-

2.2.3 Infinite Limits and Vertical Asymptotes

What does it mean for $\lim_{x \to a} f(x) = \infty$?

Example. $\lim_{x \to 2} \frac{1}{(x-2)^2} =$



If $\lim_{x \to a} f(x) = \pm\infty$ or

Example. Find all vertical asymptotes of $f(x) = \frac{8x+16}{x^2-4}$.

Example continued.

Example continued.

2.3 Calculating Limits

Learning Objectives: After completing this section, we should be able to

- calculate limits using various Limit Laws and properties.

2.3.1 Limit Laws

Suppose that c is any constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist; i.e., they are equal to a real number. Then

1. Constant multiples:

2. Sums:

3. Products:

4. Quotients:

5. Powers:

6. Roots:

Example. Suppose $\lim_{x\to a} f(x) = 2$ and $\lim_{x\to a} g(x) = -1$. Compute

$$\lim_{x\to a}\left(5\frac{f(x)}{g(x)} - (g\,(x))^4 + g(x)\sqrt{f(x)}\right).$$

2.3.2 Computing Limits

Given $f(x)$, how do we compute limits?

- If there is no bad behavior, just plug in $x = a$.

  Example.

- If there is bad behavior, attempt to tame it.

  Example.

Example. $\lim_{x \to 0} \frac{\frac{1}{5+x} - \frac{1}{5}}{x}$

Example. $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$

You try!

Example. $\lim_{x\to 3} \frac{\frac{1}{7}+\frac{1}{x-10}}{x-3}$

Summary:

1. If no bad behavior at $x = a$,

2. If bad behavior,

   (a)

   (b)

   (c)

3. If piecewise,

   (a) If

   (b) If

4. If bad behavior cannot be eliminated,

2.5 Continuity and the Intermediate Value Theorem

Learning Objectives: After completing this section, we should be able to

- define continuity and discontinuity.
- state and apply the Intermediate Value Theorem.

Definition. A function f is *continuous at* $x = a$ if

This means 3 things:

1. $\lim_{x\to a} f(x)$
2. $f(a)$
3.

Example. Consider $f(x) = \frac{x+1}{x^2-4}$.

Example. Consider $f(x) = \sqrt{x}$.

Types of discontinuities:

- Jump

$\longleftrightarrow x$

- Removable

$\longleftrightarrow x$

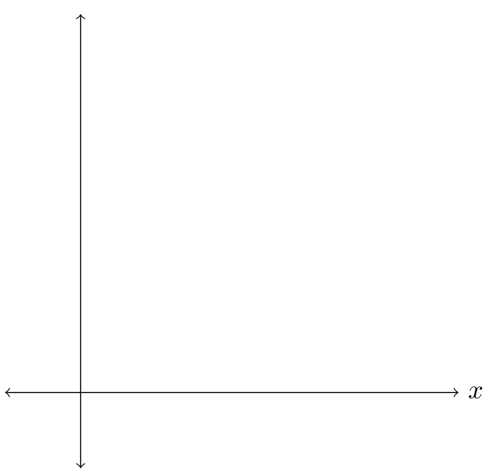
- Infinite

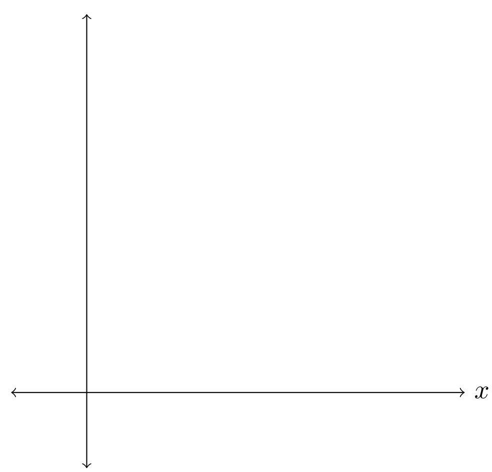
$\longleftrightarrow x$

Example. Let $f(x) = \begin{cases} x^2 - c, & \text{if } x < 5, \\ 4x + 2c, & \text{if } x \geq 5. \end{cases}$

Find c such that f is continuous.

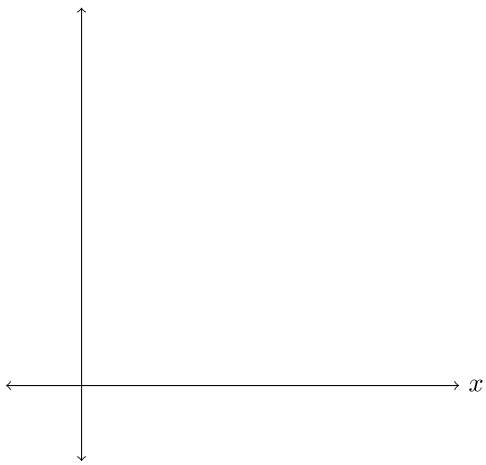
Question. *True or False: Pick any number L between $f(a)$ and $f(b)$. Then, there is an x-value c between a and b such that $f(c) = L$.*





Theorem (Intermediate Value Theorem)**.** *Assume f is continuous on $[a, b]$, and*

Sketch:



Why do we care about IVT?

Application of IVT: Root finding problems

Example. Kepler's equation for orbits (planets, satellites, etc...) is given by $y = x - a\sin(x)$ where

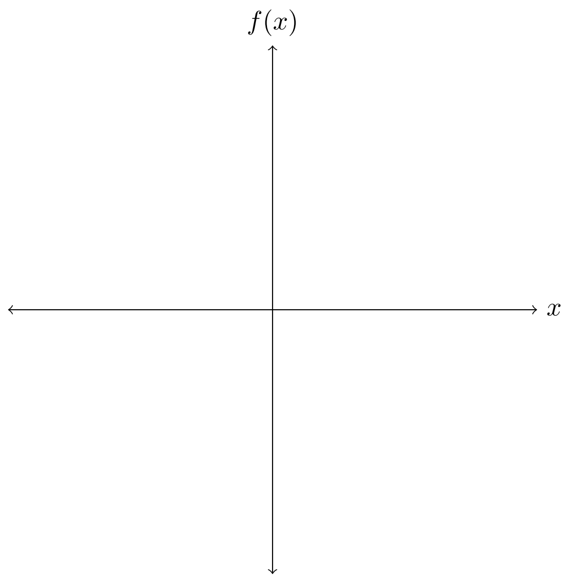
Example continued.

2.6 Limits at Infinity and Horizontal Asymptotes

Learning Objectives: After completing this section, we should be able to

- define the limits of a function at infinity and determine horizontal asymptotes of functions, if there are any.
- understand the infinite limits of a function at infinity.

Example. We've encountered the function $f(x) = \frac{8x+16}{x^2-4}$ before.



It looks maybe a horizontal asymptote too. Perhaps $y = 0$?

Definition. $x = a$ is a *vertical asymptote* if

Definition. $y = L$ is a *horizontal asymptote* if

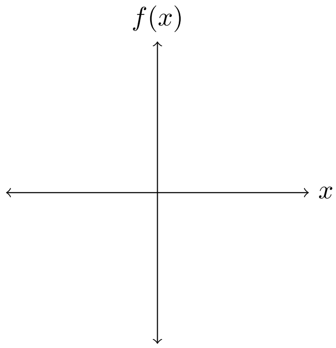
Example.

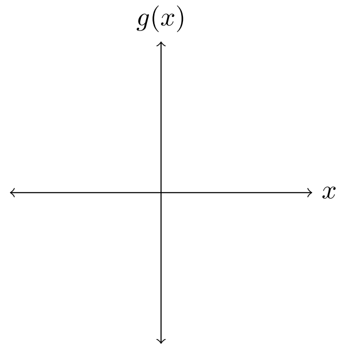
$$\lim_{x\to\infty} \frac{8x+16}{x^2-4}$$

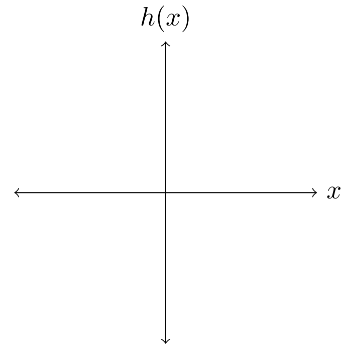
What is $\frac{\infty}{\infty}$?

Can we do some algebra to clean up $\lim_{x\to\infty} \dfrac{8x+16}{x^2-4}$ and get an actual value instead of an indeterminate form?

Question. *Is it possible to have 2 horizontal asymptotes?*







Example. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$

Question. *Is it possible to have more than 2 horizontal asymptotes?*

Question. *How many vertical asymptotes can we have?*

Consider $\lim_{x\to\infty} \frac{ax^n}{bx^m}$.

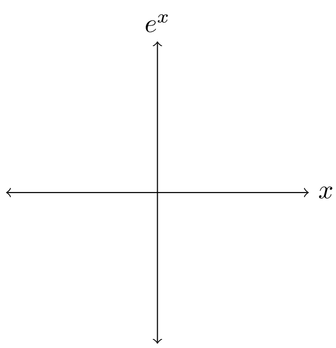
- If $n > m$, the limit is
- If $n < m$, the limit is
- If $n = m$, the limit is

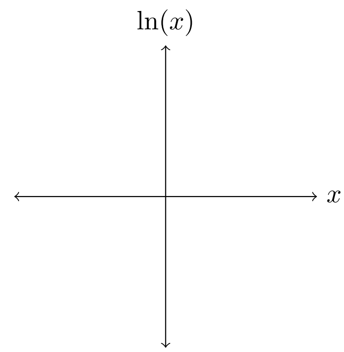
Example. $\lim_{x\to\infty} \frac{5x^5 - 6x^2 + 10^{1000}}{3x^5 + 10x^3 - 1}$

Example. $\lim_{x\to\infty} \frac{10^{100}x^5}{0.001x^{5.01}}$

Question. *Note that* $\lim_{x\to\infty} 5x+1=\infty$. *Are there any horizontal or vertical asymptotes?*

Other functions to know:



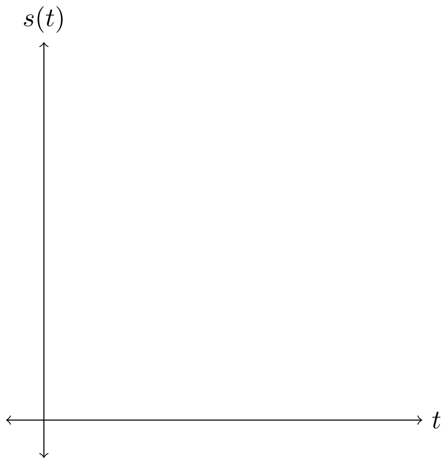


2.7 Derivatives and Rates of Change

Learning Objectives: After completing this section, we should be able to

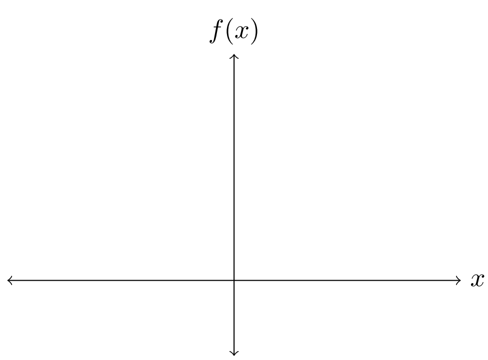
- define the slope of the tangent line to a curve at a point as the limit of the slopes of secant lines of the curve.
- define the instantaneous velocity of a moving object as the limit of its average velocity.
- establish the definition of the derivative and interpret it as the slope of the tangent line to a curve.
- interpret the derivative as the instantaneous rate of change.

Recall from earlier: If we know position $s(t)$, how do we get the instantaneous velocity at time t?



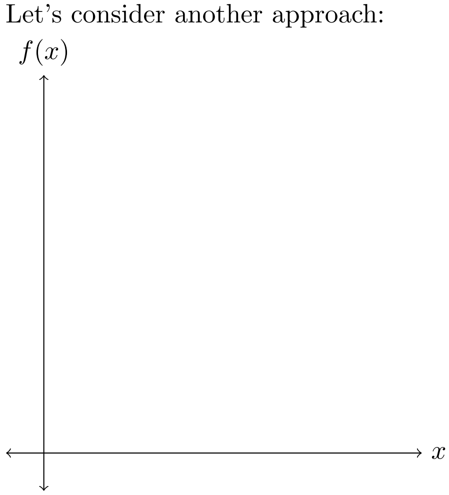
Definition. The *instantaneous rate of change* of $f(x)$ at

Example. Find the equation of the tangent line to $f(x) = x^2 + 2x + 1$ at $x = 1$.



Example continued:

Let's consider another approach:



Example. Let's find the slope of $f(x) = x^2 + 2x + 1$ at $x = 1$ again with this alternative limit.

You try!

Example. Find the equation of the tangent line $f(x) = (x - 1)^2$ at $x = 2$.

Definition. The *slope of the tangent line* at a point x is

Definition. The *derivative of* $f(x)$ is the function

What is the derivative? How do we interpret what it means?

Summary:

- The derivative at a point $x = a$ is

- The derivative at any point x is

- What does $f'(x)$ mean?
  - Instantaneous
  - Slope
  - Slope

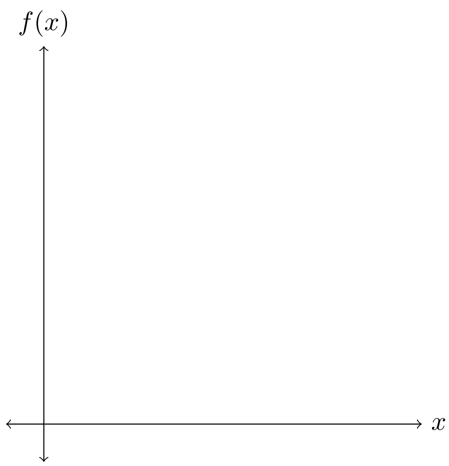
2.8 Derivative as a Function

Learning Objectives: After completing this section, we should be able to

- define and find the derivative f' as a new function derived from a function f.
- denote a derivative using Leibniz notation and prove the fact that the if a function is differentiable then it is continuous.
- analyze the cases in which a function fails to be differentiable.
- analyze whether the derivative of a function is differentiable.

Definition. Recall that the *derivative* of $f(x)$ is given by

Visually:



A common problem is finding the equation of a tangent line to a function. We need

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-

Example. Recall from last time, we found the equation of the tangent line to $f(x) = x^2 + 2x + 1$ at $x = 1$.

Example. What is the derivative of $f(x) = x^2 + 2x + 1$?

Example. Find the equation of the tangent line to $f(x) = x^2 + 2x + 1$ whose slope is 6.

You try!

Example. Find the derivative of $f(x) = \frac{1}{3x-1}$.

You try!

Example. Find the equation of the tangent line to $f(x) = \frac{1}{3x-1}$ at $x = 1$.

2.8.1 Differentiablity

Recall

Definition. The derivative of the function $f(x)$ is given by

When will we get $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ does not exist?

1. Corners

   $\longleftrightarrow x$

2. Cusps

   $\longleftrightarrow x$

3. Vertical Tangents

   $\longleftrightarrow x$

4. Discontinuities

   $\longleftrightarrow x$

Theorem. *If f is differentiable at $x = a$, then*

Proof. Assume $f(x)$ is differentiable at a.

$\square$

We proved differentiable implies continuity.

Question. *True or False: If f is continuous, then f is differentiable.*

2.8.2 Higher Order Derivatives

Since f' is a function, there is nothing stopping us from taking the derivative of f'.
Notation:

Example. Find $f''(x)$ if $f(x) = x^3 - x$.

Example Continued: